

A CLASSIFICATION OF RELATION TYPES OF ORE EXTENSIONS OF DIMENSION 5

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ABSTRACT. In order to study AS-regular algebras of dimension 5, we consider dimension 5 graded iterated Ore extensions generated in degree one. We classify the possible degrees of relations and structure of the free resolution for extensions with 3 and 4 generators. We show that every known type of algebra of dimension 5 can be realized by an Ore extension and we consider which of these types cannot be realized by an enveloping algebra.

0. INTRODUCTION

The study of Artin-Schelter (AS) regular algebras was introduced by Artin and Schelter in 1987 [AS]. AS-regular algebras correspond to noncommutative homogeneous coordinate rings of weighted projective spaces and so their classification is of interest in the field of noncommutative algebraic geometry. Our goal in this paper is to classify possible types of graded iterated Ore extensions of dimension 5 which are generated in degree one. We present an interesting example of an Ore extension with 2 degree one generators with the property that it has a Hilbert series which cannot be realized by any enveloping algebra. We also list all possible types of dimension 5 iterated Ore extensions with 3 and 4 degree one generators and consider which of these cannot be realized by the enveloping algebra of any \mathbb{N} -graded Lie algebra.

An Artin-Schelter regular algebra over K that is generated in degree one has finite presentation $A = \frac{K\langle x_1, \dots, x_b \rangle}{I}$, and its trivial module, K , has minimal free resolution:

$$(0.1) \quad 0 \rightarrow A(-l) \rightarrow A(-l+1)^b \rightarrow \bigoplus_{i=1}^n A(-l+a_i) \rightarrow \dots \\ \dots \rightarrow \bigoplus_{i=1}^n A(-a_i) \rightarrow A(-1)^b \rightarrow A \rightarrow K \rightarrow 0,$$

where any minimal generating set of I has n elements with the degree of the i th element equal to a_i . (For a proof see, for example, [Rog, Lemma 2.11 together with the discussion following Definition 2.1].) This resolution is often described via *graded Betti numbers* where $\beta_{i,j}$ is equal to the number of copies of $A(-j)$ appearing

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in the i th step of the resolution. The *Hilbert series* of A is $h_A(t) = \sum_{n \in \mathbb{N}} (\dim_K A_n) t^n$ where A_n is the n th graded piece of A . The Hilbert series can be computed directly from the free resolution: $h_A(t) = \frac{1}{q(t)}$ where $q(t) = \sum_{i,j} (-1)^i \beta_{i,j} t^j$. (See, for example, [Rog, 2.6].)

Thus, there are many invariants that we can use to discuss the possible *classification* of types of AS-regular algebras. Most generally, we can use their Hilbert series, although there are algebras with fundamentally different structures that share the same series. More refined, we can use their *relation type* (the number and degree of the relations in the minimal generating set of I , which we will denote by (a_1, \dots, a_n) where $a_1 \leq \dots \leq a_n$). More refined still, we can refer to the *resolution type*, or the set of graded Betti numbers of A . The most concrete option for classifying AS-regular algebras, and one beyond the scope of this paper, would be to list the possible *families of relations* for the algebras by explicitly writing the possible coefficients of the relations. For example, an AS-regular algebra of dimension 2 which is generated in degree one is isomorphic to $\frac{K\langle x_1, x_2 \rangle}{\langle r \rangle}$ where $r = x_2 x_1 - q x_1 x_2$, $0 \neq q$ (in which case the algebra is called the *quantum plane*) or $r = x_2 x_1 - x_1 x_2 - x_1^2$ (and the algebra is called the *Jordan plane*).

Substantial progress has been made on the classification of AS-regular algebras generated in degree one of dimension 3 and 4 [ATVdB1, ATVdB2, LPWZ, RZ, ZZ1, ZZ2]. In dimension 3, the possible families of relations are known. In dimension 4, the possible resolution types are known provided that the algebra is a domain, and under mild assumptions, the possible families of relations are known in the case where the algebras are also assumed to be \mathbb{Z}^2 -graded, i.e. $A = \frac{K\langle x_1, \dots, x_b \rangle}{I}$, $\deg(x_i) \in \{(1, 0), (0, 1)\}$ for all i , and I is homogeneous in $\mathbb{Z} \times \mathbb{Z}$. In dimension less than 5, it is known that each Hilbert series has a unique resolution type, every resolution type can be realized by the universal enveloping algebra of an \mathbb{N} -graded Lie algebra, and every resolution type can be realized by an algebra which is \mathbb{Z}^2 -graded.

More recently, Floystad and Vatne studied dimension 5 AS-regular algebras generated in degree one with 2 generators. They found an example of an AS-regular algebra for which there is no enveloping algebra with the same Hilbert series [FV, Section 4] and, under mild assumptions, provided a short list of all possible relation types, leaving open the question of whether there exist AS-regular algebras with relation type $(4, 4, 4, 5)$ or $(4, 4, 4, 5, 5)$ [Section 5]. Wang and Wu used A_∞ techniques to further classify dimension 5 algebras with 2 generators. They found several families of algebras with relation type $(4, 4, 4, 5, 5)$ and proved that there is an Ore extension of this type (although there is no enveloping algebra with this relation type) [WW, Section 5.2]. Zhou and Lu further classified the algebras under the additional assumption of a \mathbb{Z}^2 -grading and listed all possible families of relations in each case. They found there is no \mathbb{Z}^2 -graded algebra of type $(4, 4, 4, 5)$ [ZL, Proposition 5.3], although it remains an open question whether a regular algebra with this relation type exists.

To classify Ore extensions we note that, by the symmetry of the free resolution, the relation type uniquely determines the resolution type in dimension 5, although

this is probably false in higher dimensions. After introducing some preliminary definitions and a convenient presentation for Ore extensions in Section 1 and listing the possible degrees of variables of a 5-dimensional Ore extension in Section 2, we prove the following in Section 3.

Theorem 0.1 (Theorem 3.2). *There is an Ore extension with 2 degree one generators that has Hilbert series $h_A(t) = \frac{1}{(1-t)^2(1-t^2)(1-t^3)(1-t^5)}$ and relation type $(3,4,7)$.*

Together with other results in the field, this means that every known type of AS-regular algebra can be realized by an Ore extension. It remains an open question whether there is an AS-regular algebra of relation type $(4,4,4,5)$ and an Ore extension of the same type, although our preliminary calculations suggest that the latter is unlikely. In Section 4 we prove that:

Theorem 0.2 (Theorem 4.1, Theorem 4.2, Theorem 4.3). *The relation types for a dimension 5 Ore extension with 4 degree one generators are $(2,2,2,2,2)$, $(2,2,2,2,3)$, and $(2,2,2,2,3,3)$. There is no enveloping algebra with type $(2,2,2,2,3)$ although there are enveloping algebras for the other relation types.*

In Section 5, we prove:

Theorem 0.3 (Theorem 5.1, Theorem 5.2, Theorem 5.3, Theorem 5.4). *The relation types for a dimension 5 Ore extension with 3 degree one generators are $(2,3,3,3,3)$, $(2,2,3)$, and $(2,2,3,4)$. There is no enveloping algebra with type $(2,2,3)$ although there is an enveloping algebra for types $(2,2,3,4)$ and $(2,3,3,3,3)$.*

To complete the classification of Ore extensions of dimension 5, it is then natural to ask:

Question 0.4. *Is there a dimension 5 iterated Ore extension with 2 degree one generators and relation type $(4,4,4,5)$?*

Since our classification of Ore extensions was motivated by a desire to classify AS-regular algebras in general, we also would like to know:

Question 0.5. *Is there an AS-regular algebra of dimension 5 with one of the Hilbert series considered in this paper that has a different relation type than that of an Ore extension?*

Question 0.6. *Is there an AS-regular algebra of dimension 5 with a different Hilbert series than those considered in this paper?*

Both of these would help to answer our larger underlying question:

Question 0.7. *Can every type of AS-regular algebra be realized by an Ore extension?*

1. PRELIMINARIES

In this section we review basic definitions for AS-regular algebras and iterated Ore extensions and find a convenient presentation for the latter. The interested reader may find [GW, Chapters 1 and 2] a useful reference for some of the claims made about Ore extensions.

Definition 1.1. Let R be a ring. An *Ore extension* $R[x, \sigma, \delta]$ is a ring with elements of the form $f(x) = \sum_{i=0}^n a_i x^i$, $a_i \in R$ and multiplication satisfying $xr = \sigma(r)x + \delta(r)$ for all $r \in R$, where σ is an endomorphism of R and δ is a σ -derivation of R , i.e. $\delta(r_1 r_2) = \sigma(r_1)\delta(r_2) + \delta(r_1)r_2$ for all $r_1, r_2 \in R$.

An *iterated Ore extension* $R[x_1, \sigma_1, \delta_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is an Ore extension where for all $j \geq 1$, σ_j and δ_j are a ring endomorphism and a σ_j -derivation of $R_{(j-1)} := R[x_1, \sigma_1, \delta_1][x_2, \sigma_2, \delta_2] \cdots [x_{j-1}, \sigma_{j-1}, \delta_{j-1}]$, respectively. Elements in this extension have the form $\sum_{i=0}^n a_i x_1^{i_1} \cdots x_n^{i_n}$, $a_i \in R$.

We are most interested in specific types of iterated Ore extensions and will modify our definition as additional desirable properties are introduced.

We now find it convenient to review some definitions and notation. A more thorough introduction to the material, as well as a proof of Theorem 1.2, can be found in Bergman's paper [Ber, Section 1].

Suppose R is an associative algebra with unity over a field K and we have a presentation of R by a family X of generators and a family S of relations. In practice, we care about the ideal generated by S , call it I . We have $R \cong \frac{K\langle X \rangle}{I}$ where $K\langle X \rangle$ is the free associative K -algebra on $\langle X \rangle$ and $\langle X \rangle$ is the free semigroup with 1 on X . A subset, $B \subseteq S$, is a *minimal generating set* for I if no proper subset of B generates I . Fix a total ordering on $\langle X \rangle$ with the property that if $w < v$ then $uw < uv$ and $wu < vu$ for all $u \in \langle X \rangle$. Such an ordering will be called a *semigroup total ordering*. Every relation $\sigma \in S$ can be written in the form $W_\sigma = f_\sigma$ where W_σ is a monomial and is larger than any of the monomials in f_σ . We call W_σ the *leading term* (denoted LT) of the relation σ . We can assume that the leading term is always monic since K is a field. We can also take S such that all leading terms are distinct (since otherwise we could subtract a scalar multiple of one relation from another to get two relations with different leading terms which generate the same ideal).

A word w is *irreducible* under S if it does not contain any W_σ as a subword. Otherwise, w contains some W_σ , say $w = uW_\sigma v$ and we consider the K -linear *reduction* map $r_{uW_\sigma v} : K\langle X \rangle \rightarrow K\langle X \rangle$ which sends $uW_\sigma v$ to $uf_\sigma v$ and fixes all other elements of $\langle X \rangle$. A finite sequence of reductions $r_1 \cdots r_n$ ($r_i = r_{u_i W_{\sigma_i} v_i}$) is *final* on w if $r_1 \cdots r_n(w)$ is irreducible. The word w is *reduction unique* if its images under all final sequences of reductions are the same.

A 5-tuple (σ, τ, u, v, w) with $\sigma, \tau \in S$ and $u, v, w \in \langle X \rangle$ is an *overlap ambiguity* if $u, v, w \neq 1$, $W_\sigma = uv$, and $W_\tau = vw$ and an *inclusion ambiguity* if $\sigma \neq \tau$, $W_\sigma = v$, and $W_\tau = uvw$. An ambiguity is *resolvable* if there exist compositions of reductions s and s' such that $s(r_{W_\sigma w}(uvw)) = s'(r_{W_\tau}(uvw))$ (in the case of an overlap ambiguity) or $s(r_{uW_\sigma w}(uvw)) = s'(r_{W_\tau}(uvw))$ (in the case of an inclusion ambiguity). A set S of relations satisfies the *diamond condition* if all reduction ambiguities (overlap and inclusion) are resolvable, in which case we say that S is a *Gröbner basis* of I . A Gröbner basis is *reduced* if all leading terms are monic and no element in the basis has a monomial which contains the leading term of any other element in the basis. Throughout the rest of this paper, any reference to a Gröbner basis will mean a reduced Gröbner basis.

Theorem 1.2. [Ber, Theorem 1.2] *Let \leq be a semigroup total ordering having the descending chain condition and let S be a set of relations where the leading term of each relation is monic and distinct from the leading term of any other relation. Then the following conditions are equivalent:*

- (1) *S satisfies the diamond condition;*
- (2) *All elements of $K\langle X \rangle$ are reduction unique under S ;*
- (3) *A set of representatives for the elements of the algebra $R = \frac{K\langle X \rangle}{I}$ determined by the generators X and the ideal I generated by the relations S is given by the K -submodule $K\langle X \rangle_{irr}$ spanned by the S -irreducible monomials of $\langle X \rangle$.*

Continuing with definitions, we shall say that an algebra R over K is \mathbb{N} -graded if $R = \bigoplus_{n=0}^{\infty} R_n$ as K -spaces and $R_n R_m \subseteq R_{n+m}$ for all n and m . R is *connected* if $R_0 = K$. A *graded iterated Ore extension* is an iterated Ore extension as in Definition 1.1 with variables (x_1, x_2, \dots, x_n) of degrees $(deg(x_1), deg(x_2), \dots, deg(x_n))$, $deg(x_i) \geq 1$, where $\sigma_j(x_i)$ is homogeneous of degree $deg(x_i)$ and $\delta_j(x_i)$ is homogeneous of degree $deg(x_j) + deg(x_i)$ for all $n \geq j > i \geq 1$. In particular, such a ring is \mathbb{N} -graded. We refer to $(deg(x_{i_1}), \dots, deg(x_{i_n}))$ as the *degree type* of an Ore extension and require that the degrees be listed in ascending order so that the expression is unique. For the rest of this paper, any mention of an *Ore extension* will refer to a *graded iterated Ore extension* over K unless otherwise stated.

Graded lexicographic order is a total order on $\langle X \rangle$ where $w_1 > w_2$ if $deg(w_1) > deg(w_2)$ or if $deg(w_1) = deg(w_2)$ and $w_1 = a_1 a_2 \dots a_j$ comes before $w_2 = b_1 b_2 \dots b_k$ in the lexicographic order. For the rest of this paper, we will use graded lexicographic order. This has the descending chain condition. It is sometimes convenient to consider the case when the (lexicographic) order is taken to be

$$x_n > x_{n-1} > \dots > x_1,$$

and we will assume that this is the order on the variables for the rest of this section.

A finitely generated \mathbb{N} -graded Lie algebra is a Lie algebra with generators $\{x_1, \dots, x_n\}$ assigned positive degrees such that the Lie bracket preserves degree, i.e. each monomial in $[x_j, x_i]$ has degree equal to $deg(x_j) + deg(x_i)$. We note without proof that the universal enveloping algebra of an \mathbb{N} -graded finite dimensional Lie algebra L with K -basis $\{x_1, \dots, x_n\}$ can be taken to have $deg(x_1) \geq \dots \geq deg(x_n)$ and lexicographic order $x_n \geq \dots \geq x_1$ and will then have presentation $U = \frac{K\langle x_1 \dots x_n \rangle}{\langle \{r_{ji}\} \rangle}$ where for each $j > i$, there is a unique homogeneous relation r_{ji} given by

$$r_{ji} : x_j x_i = x_i x_j + \sum_{\substack{k \mid deg(x_k) = \\ deg(x_j) + deg(x_i)}} a_{ji}^k x_k, \quad a_{ji}^k \in K$$

and where the relations satisfy the diamond condition. This is a consequence of the Poincaré-Birkhoff-Witt theorem and is proven in Bergman's paper [Ber, Theorem 3.1]. The interested reader may also wish to refer to Humphrey's book on Lie algebras [Hum, Chapter 1 and Chapter 17] for the relevant background information. This statement has a converse which also holds.

Theorem 1.3. *If U has presentation $U = \frac{K\langle x_1 \cdots x_n \rangle}{\langle \{r_{ji}\} \rangle}$ where for each $j > i$, there is a unique homogeneous relation r_{ji} given by*

$$r_{ji} : x_j x_i = x_i x_j + \sum_{\substack{k \mid \deg(x_k) = \\ \deg(x_j) + \deg(x_i)}} a_{ji}^k x_k, \quad a_{ji}^k \in K$$

and where the relations satisfy the diamond condition, then U is the universal enveloping algebra of a graded Lie algebra.

Proof. Suppose an algebra U has the presentation described. Define L to be generated as a K -vector space by $\{x_1, \dots, x_n\}$ and define a multiplication on the generators of L by

$$[x_j, x_i] = \begin{cases} \sum_k a_{ji}^k x_k & j > i, \\ \sum_k -a_{ji}^k x_k & j < i, \\ 0 & j = i. \end{cases}$$

This multiplication can be extended bilinearly to general elements in L . We claim that L is the desired graded Lie algebra. The multiplication satisfies bilinearity and has the alternating property by construction. That the Jacobi identity is satisfied is equivalent to the fact that all ambiguities in U resolve by the proof of theorem 3.1 in Bergman's paper [Ber]. Finally, since this multiplication is degree preserving, L is a graded Lie algebra with enveloping algebra U . \square

Since both enveloping algebras and Ore extensions have the same basis as a weighted commutative polynomial ring with the same variables and degrees, the Hilbert series of these algebras is known: $h(t) = \frac{1}{\prod_{i=1}^n (1 - t^{\deg(x_i)})}$.

What follows are slightly modified versions of the theorems in Cohn's book Algebra Vol. 2 [Coh, Chapter 12, Theorem 1] and the proofs follow from the original proof. If R is a domain, σ a degree preserving endomorphism, and δ a degree preserving σ -derivation, then there exists an Ore extension $P = R[x, \sigma, \delta]$. Conversely we have:

Theorem 1.4. *Let R be a non-trivial graded ring and let P be a graded ring containing R with element $x \in P$ such that elements of P can be written uniquely in the form $f(x) = \sum_{i=0}^n a_i x^i$, $a_i \in R$, and satisfy a homogeneous relation $xa = \sigma(a)x + \delta(a)$ for some $\sigma(a), \delta(a) \in R$. Then σ is an endomorphism, δ a σ -derivation, and $P \cong R[x, \sigma, \delta]$ is an Ore extension.*

We now consider a presentation for (graded iterated) Ore extensions over a field K . Since we want these to be K -algebras, we are interested in the case where K is central, which means that σ_1 is the identity and δ_1 is the zero mapping. We note that the following two theorems would also hold for ungraded iterated Ore extensions with the term "homogeneous" removed from the proofs, but these are of lesser interest to us. Recall our notation established in Definition 1.1: $K_{(j)} := K[x_1] \cdots [x_j, \sigma_j, \delta_j]$.

Theorem 1.5. *If K is a field and $P = K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is a (graded iterated) Ore extension, then P has presentation*

$$P \cong \frac{K\langle x_1 \cdots x_n \rangle}{\langle \{r_{ji}\} \rangle}$$

where for each $j > i$, there is a unique homogeneous relation r_{ji} , given by

$$r_{ji} : x_j x_i = \sigma_j(x_i) x_j + \delta_j(x_i), \sigma_j(x_i) \text{ and } \delta_j(x_i) \in K_{(j-1)},$$

and these relations satisfy the diamond condition.

Proof. Let K be a field and suppose P is an iterated Ore extension: $P = K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$. Given $j > i$, $x_j x_i = \sigma_j(x_i) x_j + \delta_j(x_i)$ where σ_j is an endomorphism of $K_{(j-1)}$ and δ_j is a σ -derivation of $K_{(j-1)}$ and every monomial in the equation has degree equal to $\deg(x_j) + \deg(x_i)$. Since these relations allow any element in P to be written as a linear combination of terms of the form $kx_1^{e_1} \cdots x_n^{e_n}$, the leading term of any additional relation would be of this form, which would contradict the fact that $\{x_1^{e_1} \cdots x_n^{e_n}\}$ is a K -basis for the Ore extension P . Thus, there cannot be any additional relations so each r_{ji} is unique, all reduction ambiguities must resolve, and the diamond condition is satisfied. \square

We can also prove a converse:

Theorem 1.6. *If K is a field and $P = \frac{K\langle x_1 \cdots x_n \rangle}{\langle \{r_{ji}\} \rangle}$ where for each $j > i$, there is a unique homogeneous relation r_{ji} given by $x_j x_i = \sigma_j(x_i) x_j + \delta_j(x_i)$, $\sigma_j(x_i)$ and $\delta_j(x_i) \in \frac{K\langle x_1, \dots, x_{j-1} \rangle}{\langle \{r_{ji}\} \rangle}$, and these relations satisfy the diamond condition, then $P \cong K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is a (graded iterated) Ore extension.*

Proof. Assume that we have the unique relations $\{r_{ji}\}$ satisfying the diamond condition and for the purpose of induction assume that

$$R = \frac{K\langle x_1 \cdots x_{m-1} \rangle}{\langle \{r_{ji}\} \rangle} \cong K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_{m-1}, \sigma_{m-1}, \delta_{m-1}]$$

is an Ore extension. Then any monomial in $\frac{K\langle x_1 \cdots x_m \rangle}{\langle \{r_{ji}\} \rangle}$ has the form

$x_m^{c_1} s_1 x_m^{c_2} s_2 \cdots x_m^{c_q} s_q$, $s_i \in R$ where by induction each s_i can be taken to have the form $kx_1^{f_1} \cdots x_{m-1}^{f_{m-1}}$ since $\{x_1^{e_1} \cdots x_{m-1}^{e_{m-1}}\}$ is a basis for R . By repeated application of the relations $x_m x_i = \sigma_m(x_i) x_m + \delta_m(x_i)$, the monomial can be written in the form $s'_a x_m^a + \cdots + s'_1 x_m + s'_0$, $s'_i \in R$. This representation is unique since the $\{r_{ji}\}$ satisfy the diamond condition by assumption. Thus $\{x_1^{e_1}, \dots, x_m^{e_m}\}$ is a K -basis by Theorem 1.2. Since the relations were also chosen to be homogeneous, $R[x_m, \sigma_m, \delta_m]$ is an Ore extension by Theorem 1.4 and by induction, $P \cong K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is an Ore extension. \square

Since Ore extensions for which σ_j is an automorphism for all $j \geq 1$ have especially nice properties, we also find the following result helpful:

Theorem 1.7. *In a (graded iterated) Ore extension $K[x_1, \sigma_1, \delta_1] \cdots [x_n, \sigma_n, \delta_n]$, for any $j \geq 1$, if σ_j is injective then it is an automorphism of $K_{(j-1)}$.*

Proof. Let $K_{(j-1)}^i$ denote the i th graded piece of $K_{(j-1)}$, i.e. the set of all degree i homogeneous polynomials in $K_{(j-1)}$. $K_{(j-1)}^i$ has finite K -basis $\{x_1^{f_1} x_2^{f_2} \cdots x_{j-1}^{f_{j-1}} \mid f_1 \deg(x_1) + f_2 \deg(x_2) + \cdots + f_{j-1} \deg(x_{j-1}) = i\}$. By the definition of a graded iterated Ore extension, we know that σ_j preserves degree on the generators and hence on all of $K_{(j-1)}$. Any injective map from a finite-dimensional vector space to itself must also be surjective by the rank-nullity theorem, so for any i and j , $\sigma_j|_{K_{(j-1)}^i} : K_{(j-1)}^i \rightarrow K_{(j-1)}^i$ is bijective, and thus σ_j is an automorphism of $K_{(j-1)}$ for all j . \square

Recall that we are interested in the study of Ore extensions because they provide examples of AS-regular algebras. We provide the definition of such algebras here, but readers looking for a more detailed introduction to the material may wish to refer to the notes found in [Rog, Lecture 1 and Lecture 2].

Definition 1.8. A connected graded algebra $A = \bigoplus_{i=0} A_i$ is *Artin-Schelter regular* of dimension d if

- (1) A has finite global dimension d ;
- (2) A has finite Gelfand-Kirillov dimension;
- (3) A is AS-Gorenstein, i.e.

$$\mathrm{Ext}_A^i(K, A) = \begin{cases} 0 & i \neq d \\ K(l) & i = d \end{cases}$$

where $K(l)$ is a shifted copy of K satisfying $K(l)_n = K_{l+n}$.

It is a fact that the universal enveloping algebra of a graded Lie algebra is Artin-Schelter regular [FV, Theorem 2.1]. From the presentations provided, it is also clear that any such enveloping algebra is a specific example of an Ore extension.

More generally, it is also known that Ore extensions where σ_j is an automorphism for all $j \geq 1$ are Artin-Schelter regular (see [AST, Proposition 2]).

Motivated by the study of AS-regular algebras, our goal in the rest of this paper is to classify the possible relation and resolution types of all dimension 5 “Ore extensions,” by which we mean “graded iterated Ore extensions with injective (and thus bijective) σ'_j s, generated in degree one.” For the sake of brevity, we will wish to have any easy way to refer to such algebras.

Definition 1.9. An *AS-Ore* extension is a graded iterated Ore extension with σ_j injective for every $j \geq 1$ and which is generated in degree one as a K -algebra.

We note that this definition is in no way standard (and in particular, there are AS-regular algebras that are not generated in degree one and which we have chosen not to study at this time). We also note that the enveloping algebra of an \mathbb{N} -graded Lie algebra which is generated in degree one is also an AS-Ore extension.

2. A CLASSIFICATION OF POSSIBLE DEGREE TYPES OF AS-ORE EXTENSIONS

Our goal in this section is to list the 7 possible degree types for an Ore extension generated in degree one with 5 variables. We note that when considering fully general Ore extensions, we may either order the variables by descending degree ($\deg(x_1) \leq \cdots \leq \deg(x_n)$) at the expense of fully controlling the lexicographic

order in the Ore extension $K[x_{i_1}] \cdots [x_{i_n}, \sigma_{i_n}, \delta_{i_n}]$, or we may assume that $x_5 > \cdots > x_1$ in the lexicographic order at the expense of controlling the degrees of these variables. We transfer freely between these two conventions depending on which is more convenient in each situation and the convention we use does not affect the validity of any theorems we prove for general extensions.

Lemma 2.1. *If $A = K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is an AS-Ore extension and $\deg(x_k) \neq 1$ then there exist i and j with $\deg(x_i) + \deg(x_j) = \deg(x_k)$.*

Proof. Assume $\deg(x_k) > 1$. If the algebra is generated in degree one, $x_k = f(\hat{X})$ where $\hat{X} = \{x \in X | \deg(x) = 1\}$, $X = \{x_1, \dots, x_n\}$, and $f(\hat{X})$ is a (noncommutative) polynomial in variables from \hat{X} . This gives the relation $0 = x_k - f(\hat{X})$. By Theorem 1.5, $A \cong \frac{K\langle x_1, \dots, x_n \rangle}{I}$ where $I = \langle \{r_{ji}\}_{j>i} \rangle$, so any relation is generated by the $\{r_{ji}\}$ and we have that, in the free algebra, $x_k - f(\hat{X}) = \sum_{i,j} p_{ji}(X) r_{ji} q_{ji}(X)$ where $p_{ji}(X)$ and $q_{ji}(X)$ are (noncommutative) polynomials in X and the r_{ji} are the generators of I and hence have degree greater than zero.

Equating polynomials, we find that the monomial x_k must appear in the right side of this equation, so there exist fixed i and j and monomials m_p , m_r , and m_q of p_{ji} , r_{ji} , and q_{ji} with $x_k = m_p m_r m_q$. Since $\deg(m_r) > 0$, we get that m_p and m_q must be scalars and ax_k is a monomial of r_{ji} where $0 \neq a \in K$. So r_{ji} is a relation with leading term $x_j x_i$ and has now been shown to have a scalar multiple of x_k as a term. Since r_{ji} is also homogeneous, this means that $\deg(x_i) + \deg(x_j) = \deg(x_k)$. \square

Corollary 2.2. *There is no AS-Ore extension with Hilbert series*

$$h(t) = \frac{1}{(1-t)^k \prod_{j=1}^{n-k} (1-t^{i_j})}, \quad i_j > 2 \text{ for all } j, \quad k < n.$$

Proof. Suppose this is possible. Choose k such that $\deg(x_k)$ is minimal amongst variables with degree greater than 1. By Lemma 2.1, $\deg(x_k) = \deg(x_i) + \deg(x_j)$ for some i and j . If $\deg(x_i) = \deg(x_j) = 1$ then this equation says $\deg(x_k) = 2$, but an Ore extension with the given Hilbert series cannot have any variables of degree two. Otherwise, we can assume that $\deg(x_i) > 1$. Since x_k was chosen to have smallest degree greater than 1, the equation now becomes $\deg(x_k) = \deg(x_i) + \deg(x_j) \geq \deg(x_k) + 1$, which is impossible. Thus, no Ore extension generated in degree one can have this Hilbert series. \square

Lemma 2.3. *There is no AS-Ore extension with Hilbert series*

$$h(t) = \frac{1}{(1-t)^2(1-t^2)^2 \prod_{j=1}^{n-4} (1-t^{i_j})}, \quad i_j \geq 2 \text{ for all } j.$$

Proof. To find a contradiction, assume that such an extension, A , exists. By Theorem 1.5, there exists an ordering on the variables such that $A \cong \frac{K\langle x_1, \dots, x_n \rangle}{\langle \{r_{ji}\} \rangle}$. Let x_n and x_{n-1} be degree one variables and let x_{n-2} and x_{n-3} be distinct degree

two variables in $K\langle x_{n-1}, x_n \rangle$ and so of the form

$$\begin{aligned} x_{n-2} &= a_1 x_{n-1}^2 + a_2 x_{n-1} x_n + a_3 x_n x_{n-1} + a_4 x_n^2, \\ x_{n-3} &= b_1 x_{n-1}^2 + b_2 x_{n-1} x_n + b_3 x_n x_{n-1} + b_4 x_n^2. \end{aligned}$$

Without loss of generality, we can assume that $x_n > x_{n-1}$ in the ordering. Also, we must have that $x_{n-2} < x_n$ and $x_{n-3} < x_n$ since an Ore extension cannot have x_{n-2} or x_{n-3} as the leading term of a relation. Since an Ore extension has no leading term of the form x_n^2 and a unique term of the form $x_n x_{n-1}$, we get that $a_4 = b_4 = 0$ and one of a_3 and b_3 is 0. Without loss of generality, assume that $b_3 = 0$. Then the relation $x_{n-3} = b_1 x_{n-1}^2 + b_2 x_{n-1} x_n$ has a leading term inconsistent with an Ore extension. \square

Theorem 2.4. *For an AS-Ore extension with 5 variables, one of the following options represents the possible degree type of the extension.*

- (1) (1, 1, 2, 3, 5),
- (2) (1, 1, 2, 3, 4),
- (3) (1, 1, 2, 3, 3),
- (4) (1, 1, 1, 2, 3),
- (5) (1, 1, 1, 2, 2),
- (6) (1, 1, 1, 1, 2),
- (7) (1, 1, 1, 1, 1).

Proof. Clearly an Ore extension with no variables of degree one cannot be generated in degree one.

Similarly there is no Ore extension generated in degree one with just 1 degree one variable. For if there were such an Ore extension and x_k were of minimal degree amongst the remaining 4 variables, Lemma 2.1 says that $\deg(x_k) = \deg(x_i) + \deg(x_j) \geq 1 + \deg(x_k)$ and such an inequality is impossible.

If the Ore extension has exactly 2 degree one variables then Corollary 2.2 implies that there is at least one variable of degree two. If x_k is of minimal degree amongst the remaining 2 variables then Lemma 2.1 tells us that $\deg(x_k) \in \{2, 3\}$ since these are the only possible combinations of $\deg(x_i) + \deg(x_j)$. By Lemma 2.3, there can be at most one variable of degree two, so $\deg(x_k) = 3$. Again by Lemma 2.1, the final and largest degree variable, x_l , satisfies $\deg(x_l) = \deg(x_i) + \deg(x_j)$ for some i and j so $\deg(x_l) \in \{3, 4, 5\}$. Thus, the list of possible degree types for an Ore extension with exactly 2 degree one variables is

- (1) (1, 1, 2, 3, 5),
- (2) (1, 1, 2, 3, 4),
- (3) (1, 1, 2, 3, 3).

If the Ore extension has exactly 3 degree one variables then it must have at least one degree two variable and the remaining variable, by Lemma 2.1, must be of degree two or three. Thus, the list of possible degree types in this case is

- (4) (1, 1, 1, 2, 3),
- (5) (1, 1, 1, 2, 2).

If the Ore extension has exactly 4 degree one variables, then Lemma 2.1 tells us that the remaining variable must be degree two and the possible degree type is

- (6) (1, 1, 1, 1, 2).

Finally, it is possible for the Ore extension to have 5 degree one variables and degree type

$$(7) (1, 1, 1, 1, 1).$$

□

While this result technically only restricts the possible degree types of AS-Ore extensions, it is also true that, for each of the 7 possible options listed, there exists an AS-Ore extension with the given type. A commutative ring in five variables is an example of an algebra with type $(1,1,1,1,1)$. For options 2-6, there are enveloping algebras with variables of appropriate degrees (see [FV, Section 3], Theorem 4.3, Theorem 5.2, Theorem 5.4). Finally, we construct an AS-Ore extension with degree type $(1,1,2,3,5)$ in the next section (Theorem 3.2).

3. AN AS-ORE EXTENSION WITH DEGREE TYPE $(1, 1, 2, 3, 5)$

For AS-regular algebras of dimension at most 4, it is known that every Hilbert series has a unique relation type and every relation type can be realized by the enveloping algebra of a graded Lie algebra. In their paper, Floystad and Vatne asked whether this held in dimension 5 and constructed, as a counter example, an AS-regular algebra with 2 degree one generators and Hilbert series $h(t) = \frac{1}{(1-t)^2(1-t^2)(1-t^3)(1-t^5)}$. By looking at the shifts in the free resolution, they prove that there is no enveloping algebra of a graded Lie algebra with this Hilbert series [FV, Proposition 3.4 and Theorem 4.2]. Based on the presentation of an enveloping algebra given above, we provide an alternate proof of this result.

Proposition 3.1. *There is no enveloping algebra of a graded Lie algebra which is generated in degree one with $h(t) = \frac{1}{(1-t)^2(1-t^2)(1-t^3)(1-t^5)}$.*

Proof. Assume to the contrary that there is such an algebra. Then there are variables $(x_5, x_4, x_3, x_2, x_1)$ with respective degrees $(1, 1, 2, 3, 5)$. Consider the possible terms in the relations. We will list only those required to show the contradiction.

$$r_{54} : x_5x_4 = x_4x_5 + a_1x_3$$

$$r_{52} : x_5x_2 = x_2x_5$$

$$r_{42} : x_4x_2 = x_2x_4$$

$$r_{32} : x_3x_2 = x_2x_3 + b_1x_1.$$

Here a_1 and b_1 must be nonzero for this algebra to be generated in degree one (by the proof of Lemma 2.1) since these are the only relations of degree two and five respectively. Additionally, the middle two relations cannot contain any additional terms since they are degree four and this algebra contains no variable of degree exactly 4. Now consider:

$$\begin{aligned} x_5(x_4x_2) &= x_5x_2x_4 \\ &= x_2x_5x_4 \\ &= x_2x_4x_5 + a_1x_2x_3, \text{ while} \\ (x_5x_4)x_2 &= x_4x_5x_2 + a_1x_3x_2 \\ &= x_4x_2x_5 + a_1x_2x_3 + a_1b_1x_1 \\ &= x_2x_4x_5 + a_1x_2x_3 + a_1b_1x_1. \end{aligned}$$

In order for this overlap to resolve, $a_1b_1 = 0$, which is impossible if this algebra is generated in degree one. \square

It is natural to ask whether every relation type can be realized by a generalization of an enveloping algebra, in particular an AS-Ore extension. In the literature for algebras of dimension 5 there are currently only two known relation types that cannot be realized by an enveloping algebra. One has Hilbert series $h(t) = \frac{1}{(1-t)^2(1-t^2)(1-t^3)^2}$, relation type $(4, 4, 4, 5, 5)$, and can be realized by an AS-Ore extension [WW, Section 5.2].

In support of the hypothesis that all relation types can be realized by an AS-Ore extension, we present an example of an AS-Ore extension with the other known relation type that cannot be realized by an enveloping algebra. This is equivalent to finding an example of an AS-Ore extension with the appropriate Hilbert series since Floystad and Vatne have already classified the possible relation types of algebras with two generators [FV, Theorem 5.6], and the relation type of an algebra with this Hilbert series is unique.

The following example was found by writing the general relations provided by Theorem 1.5 and using the mathematical software program Mathematica to solve the large system of equations that result from setting overlap ambiguities equal to 0. This proved to be an overwhelming project for the computer and some coefficients were ultimately assumed to be 0 to make the computations possible, as our goal was to prove the existence of such an algebra rather than to completely classify the possible families of relations.

Theorem 3.2. *The following relations define an AS-Ore extension (an iterated Ore extension which is graded, generated in degree one, and has each σ_i an injection) which has*

$$h(t) = \frac{1}{(1-t)^2(1-t^2)(1-t^3)(1-t^5)} \text{ and relation type } (3, 4, 7):$$

$$r_{21} : x_2x_1 = -x_1x_2$$

$$r_{32} : x_3x_2 = x_1 + bx_2x_3$$

$$r_{31} : x_3x_1 = -x_1x_3$$

$$r_{43} : x_4x_3 = x_2 + bx_3x_4$$

$$r_{42} : x_4x_2 = b^2x_2x_4$$

$$r_{41} : x_4x_1 = x_1x_4$$

$$r_{54} : x_5x_4 = x_3 + x_4x_5$$

$$r_{53} : x_5x_3 = -x_3x_5$$

$$r_{52} : x_5x_2 = -x_2x_5 - b^2x_3x_3$$

$$r_{51} : x_5x_1 = x_1x_5 + cx_3x_3x_3,$$

where $b = e^{\frac{4\pi i}{3}}$ and $c = \frac{2b^2}{1-b+b^2}$.

Proof. Let the degrees of $(x_5, x_4, x_3, x_2, x_1)$ be $(1, 1, 2, 3, 5)$ with lexicographic order $x_5 > \dots > x_1$ so that the leading terms are as presented.

We note that, for the given degrees of these variables, each of the above relations is homogeneous. To check that this is an Ore extension, we then check that all reduction ambiguities resolve. All ambiguities have the form $x_kx_jx_i$ where $k > j > i$

and there are a total of 10 such ambiguities for this set of relations. A computation shows that, for the given choice of b and c , all overlaps resolve. We carry out this computation in Mathematica and the code for these and future computations can be found on the author's website [Ell, Section 1]. Thus, $\{x_1^{e_1} \cdots x_n^{e_n}\}$ is a basis for this algebra and this is Ore by Theorem 1.6.

It remains to check that this is generated in degree one and that σ_j is injective for all j . To see that this algebra is generated in degree one, note that r_{32} , r_{43} , and r_{54} can be solved for x_1 , x_2 , and x_3 respectively and so everything may be expressed in terms of the degree one generators x_4 and x_5 . Note that for all $1 \leq i < j \leq 5$, $\sigma_j(x_i) = a_{ji}x_i$ where a_{ji} is a root of unity. Thus, $\sigma_j^{n_j}$ is the identity map for some n_j and so each homomorphism is injective. Thus, this algebra is an AS-Ore extension.

That the algebra has the desired Hilbert series is now immediate from the fact that it has the same basis, and therefore the same Hilbert series, as the weighted commutative polynomial ring with the same variables and degrees. We again note that the relation type of an algebra with this Hilbert series is known to be (3,4,7) [FV, Theorem 5.6], although the computations proving it in this case are also included in the online code. \square

4. A CLASSIFICATION OF RELATION TYPES FOR AS-ORE EXTENSIONS WITH 4 DEGREE ONE GENERATORS

We now begin the process of attempting to classify all possible resolution types of dimension 5 AS-Ore extensions generated in degree one, beginning with the case where the algebra has 4 degree one generators. It will be convenient to alternate between thinking of the algebra as an Ore extension with the presentation given by

Theorem 1.5, $A \cong \frac{K\langle x_1, \dots, x_n \rangle}{\langle \{r_{ji}\} \rangle}$, and as an algebra presented in terms of its degree one generators, $A \cong \tilde{A} = \frac{K\langle x_1, \dots, x_b \rangle}{I}$. We will fix the notation that A refers to

the algebra viewed as an AS-Ore extension presented by 5 generators and that \tilde{A} is an algebra isomorphic to A , viewed as generated in degree one. We get it from A by changing the ordering on the variables so that $x_i > x_j$ whenever $\deg(x_i) > 1$ and $\deg(x_j) = 1$, making a choice that allows us to solve the Ore relations for all variables that are not degree one, and writing the remaining relations in terms of the degree one generators. Changing the ordering of the variables may change the Gröbner basis of the algebra, but will of course not change the Hilbert series. By the construction of \tilde{A} , we also note that its minimal generating set cannot contain more elements of a particular degree than what the minimal generating set of A has.

By Equation (0.1), the free resolution of any dimension 5 regular algebra generated in degree one is

$$0 \rightarrow A(-l) \rightarrow A(-l+1)^b \rightarrow \bigoplus_{i=1}^n A(-l+a_i) \rightarrow \bigoplus_{i=1}^n A(-a_i) \rightarrow A(-1)^b \rightarrow A \rightarrow K \rightarrow 0$$

where b represents the number of degree one generators, l the total shift of the resolution, n the number of relations in the minimal generating set of the ideal I , and a_i the homogeneous degree of the i th relation of a fixed minimal generating set of I . By the symmetry of this free resolution, the resolution type is uniquely

determined by the a_i together with the Hilbert series of the algebra since the series will determine the value of l . Thus, it suffices to classify the possible relation types (a_1, \dots, a_n) , $a_i \leq \dots \leq a_n$, of dimension 5 AS-Ore algebras.

Recall by Theorem 2.4 that an AS-Ore extension, A , with 5 variables and 4 degree one generators will have degree type $(1,1,1,1,2)$. Since the relations for an Ore extension come from the $\{r_{ji}\}$ as described in Theorem 1.5 (and there are no additional relations since overlaps resolve by the same theorem), A will have 6 degree two relations in the Gröbner basis, 4 relations of degree three, and no relations of degree four or larger.

Let \tilde{A} denote the same algebra, A , viewed as an algebra generated in degree one via the process explained above where 1 degree two relation of A will be used to express the degree two variable in terms of the generators and the remaining 5 will be part of the minimal generating set of \tilde{A} . Since A has 4 degree three relations, \tilde{A} will have at most 4 degree three relations. It is possible that \tilde{A} will have fewer than 4 K -independent degree three relations or for these relations to be consequences of overlaps that fail to resolve rather than part of the minimal generating set. Since A has no relations of degree more than three, any relations of degree more than three in \tilde{A} must be consequences of overlaps that fail to resolve and so not part of the minimal generating set of \tilde{A} . Thus, the only candidates for the relation type of an AS-Ore extension with degree type $(1,1,1,1,2)$ are:

- (1) $(2,2,2,2,2)$,
- (2) $(2,2,2,2,3)$,
- (3) $(2,2,2,2,3,3)$,
- (4) $(2,2,2,2,3,3,3)$,
- (5) $(2,2,2,2,3,3,3,3)$.

In the next theorems, we will classify all possible relation types of an AS-Ore extension with the given degree type. We prove that types (4) and (5) above are impossible, that types (1) and (3) can be realized by enveloping algebras, and that type (2) can be realized by an AS-Ore extension but not by an enveloping algebra. This differs slightly from the comment in [FV, Section 3] where examples of enveloping algebras of types (1) and (3) are explicitly presented but the reader is encouraged to also check that there is an example of an enveloping algebras of type (2).

In order to prove that certain relation types are impossible, we need to know more about the specific leading terms and the overlaps that come from the degree two relations. We use a simplified version of Hilbert driven Gröbner basis computation, a technique used by Rogalski and Zhang to study \mathbb{Z}^2 -graded dimension 4 algebras with 3 generators [RZ] and later used by Zhou and Lu to classify possible families of relations of \mathbb{Z}^2 -graded dimension 5 algebras with 2 generators [ZL].

Let A be an AS-Ore extension with degree type $(1,1,1,1,2)$. Then A has Hilbert series $h_A(t) = \frac{1}{(1-t)^4(1-t^2)} = 1 + 4t + 11t^2 + 24t^3 + O(t^4)$. The idea of Hilbert driven basis computation is to construct \tilde{A} by viewing it as a free algebra on its degree one generators modulo an ideal I , and to identify the generators of I by comparing the Hilbert series of the constructed algebra against the known Hilbert series. In order to do this one dimension at a time, we will use a *monomial algebra* which we get by replacing each relation in the Gröbner basis with just the leading

term of the relation. The details of this construction can be found in [ZL, Section 2], along with the proof that the monomial algebra will have the same Hilbert series as the original algebra [ZL, Lemma 2.1]. Let \tilde{A}_0 denote the free algebra on four generators. Then $h_{\tilde{A}_0}(t) = 1 + 4t + 16t^2 + 64t^3 + O(t^4)$ and $h_{\tilde{A}_0}(t) - h_{\tilde{A}}(t) = 5t^2 + O(t^3)$. Thus, I must contain 5 degree two relations. Although we already knew this, the method can be used to find the number of relations in the basis of higher degrees, although the analysis does depend on which leading terms we choose for our relations.

Let x_1 be the degree two variable and without loss of generality, list the degree one variables so that $x_2 < x_3 < x_4 < x_5$. The degree two LT's (leading terms) in \tilde{A} must come from the $\{r_{ji}\}$ relations of the Ore extension, A , and so must belong to the list $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_3, x_5x_2\}$. If the set of degree two LTs in I is $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_3\}$, then let us denote the monomial algebra which has these leading terms as its relations by \tilde{A}_2 since the Hilbert series agrees with that of \tilde{A} up to dimension 2. Then $h_{\tilde{A}_2} - h_{\tilde{A}} = 4t^3 + O(t^4)$ so there must be 4 degree three relations in the Gröbner basis. The calculations for the difference of these Hilbert series were done in Mathematica, and the code is available online [Ell, Section 2].

If instead we start with LTs in I $\{x_3x_2, x_4x_3, x_5x_4, x_5x_3, x_5x_2\}$ or $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_2\}$, then $h_{\tilde{A}_2} - h_{\tilde{A}} = 3t^3 + O(t^4)$ and there are 3 degree three relations in the basis. If we start with LTs $\{x_4x_3, x_4x_2, x_5x_4, x_5x_3, x_5x_2\}$, $\{x_3x_2, x_4x_2, x_5x_4, x_5x_3, x_5x_2\}$, or $\{x_3x_2, x_4x_3, x_4x_2, x_5x_3, x_5x_2\}$ then there will be 2 degree three relations by a similar analysis.

So we have found that the Gröbner basis of an AS-Ore extension with the given degree type has 5 degree two relations and 2-4 degree three relations. We could continue the process to see what the possible LTs of degree three are and if the basis has additional relations of higher degree, but it is not useful for our analysis. We remain more interested in the number and degrees of the minimal generators since these completely classify the possible relation types of the algebra, and we know the algebra has no minimal relations of degree greater than three. We still need to investigate, in each case, which of the degree three relations are part of the minimal generating set and which are simply consequences of overlaps that fail to resolve.

Theorem 4.1. *There is no AS-Ore extension with degree type $(1, 1, 1, 1, 2)$ and minimal relation type $(2, 2, 2, 2, 2, 3, 3, 3, 3)$.*

Proof. Assume to the contrary that there is such an AS-Ore extension $A = K[x_{i_1}] \cdots [x_{i_5}, \sigma_{i_5}, \delta_{i_5}]$. Label the degree two variable x_1 and label the degree one variables so that $x_2 < x_3 < x_4 < x_5$ in the ordering. (We make no assumption about when x_1 is adjoined.) The list of reduced degree 2 monomials is

$$\{x_1, x_2x_2, x_2x_3, x_3x_3, x_2x_4, x_3x_4, x_4x_4, x_2x_5, x_3x_5, x_4x_5, x_5x_5\}.$$

From Theorem 1.5, x_j must occur only to the first power in the relation with leading term x_jx_i and x_k should not appear for any $x_k > x_j$. Based on these observations,

we will write the most general possible degree two relations:

$$\begin{aligned}
r_{32} : x_3x_2 &= b_1x_1 + b_2x_2x_2 + b_3x_2x_3 \\
r_{42} : x_4x_2 &= e_1x_1 + e_2x_2x_2 + e_3x_2x_3 + e_4x_2x_4 + e_6x_3x_3 + e_7x_3x_4 \\
r_{43} : x_4x_3 &= d_1x_1 + d_2x_2x_2 + d_3x_2x_3 + d_4x_2x_4 + d_6x_3x_3 + d_7x_3x_4 \\
r_{52} : x_5x_2 &= i_1x_1 + i_2x_2x_2 + i_3x_2x_3 + i_4x_2x_4 + i_5x_2x_5 + i_6x_3x_3 + i_7x_3x_4 + i_8x_3x_5 \\
&\quad + i_9x_4x_4 + i_{10}x_4x_5 \\
r_{53} : x_5x_3 &= h_1x_1 + h_2x_2x_2 + h_3x_2x_3 + h_4x_2x_4 + h_5x_2x_5 + h_6x_3x_3 + h_7x_3x_4 \\
&\quad + h_8x_3x_5 + h_9x_4x_4 + h_{10}x_4x_5 \\
r_{54} : x_5x_4 &= g_1x_1 + g_2x_2x_2 + g_3x_2x_3 + g_4x_2x_4 + g_5x_2x_5 + g_6x_3x_3 + g_7x_3x_4 \\
&\quad + g_8x_3x_5 + g_9x_4x_4 + g_{10}x_4x_5.
\end{aligned}$$

(It is worth noting that if x_1 is adjoined late, some of these coefficients must be zero, although this fact will not be needed to complete the contradiction. For example, if the Ore extension is $K[x_2][x_3, \sigma_3, \delta_3][x_1, \sigma_1, \delta_1][x_4, \sigma_4, \delta_4][x_5, \sigma_5, \delta_5]$, then b_1 must be zero since $x_3x_2 = \sigma_3(x_2)x_3 + \delta_3(x_2)$ where δ_3 is a derivation of $K[x_2]$ and thus cannot map x_2 to a term containing x_1 .)

Without loss of generality, we can solve for x_1 using the relation with smallest leading term that has a nonzero coefficient of x_1 . For example, if $b_1 \neq 0$ then we can solve r_{32} to find that $x_1 = \frac{1}{b_1}(x_3x_2 - b_2x_2x_2 - b_3x_2x_3)$. Consider now the

Gröbner basis of the algebra \tilde{A} , viewed as an algebra generated in degree one.

The preceding analysis shows that the list of degree two LTs of \tilde{A} must be $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_3\}$ since this is the only set of LTs that has 4 degree three relations in the Gröbner basis. In particular, x_5x_2 is not a leading term. But if any of b_1, e_1 , or d_1 are non-zero, then x_1 can be expressed in terms of monomials smaller than x_5x_2 and the LT of r_{52} will be x_5x_2 , a contradiction. We may therefore assume that $b_1 = e_1 = d_1 = 0$ and we may assume that i_1 is not zero since again this would otherwise make the leading term of r_{52} equal to x_5x_2 . Thus \tilde{A} is obtained by using r_{52} to write $x_1 = \frac{1}{i_1}(x_5x_2 - i_{10}x_4x_5 - \cdots - i_2x_2x_2)$ and substituting this expression for x_1 in the other relations.

We are interested in the coefficient of $x_5x_2x_3$ in the reduction of the overlap $x_5x_3x_2$ in \tilde{A} since this will provide the contradiction. We compute:

$$\begin{aligned}
x_5(x_3x_2) &= x_5(0x_1 + b_3x_2x_3 + [\text{smaller terms}]) \\
&= b_3x_5x_2x_3 + [\text{smaller terms}], \text{ while} \\
(x_5x_3)x_2 &= (h_1x_1 + h_{10}x_4x_5 + [\text{smaller terms}])x_2 \\
&= \left(\frac{h_1}{i_1}(x_5x_2 - i_{10}x_4x_5 - [\text{smaller terms}]) + h_{10}x_4x_5 + [\text{smaller terms}]\right)x_2 \\
&= \frac{h_1}{i_1}x_5x_2x_2 + [\text{smaller terms}].
\end{aligned}$$

Note that $x_5x_2x_3$ is a reduced word with respect to the degree two relations of \tilde{A} . Thus $x_5(x_3x_2) - (x_5x_3)x_2 = b_3x_5x_2x_3 + [\text{smaller terms}]$ and $b_3 = 0$ if this overlap resolves. However, if $b_3 = 0$ then r_{32} becomes

$$x_3x_2 = \sigma_3(x_2)x_3 + \delta_3(x_2) = 0x_2x_3 + 0x_1 + b_2x_2x_2.$$

This would suggest that $\sigma_3(x_2) = 0$ and σ_3 is not injective, which contradicts the claim that the initial algebra was an AS-Ore extension. Thus, b_3 is not zero, this overlap does not resolve, at least one of the degree three relations in the Gröbner basis of this algebra is a consequence of an overlap, and so there are not 4 degree three relations in the minimal generating set. \square

With a little more work, the same technique can be used to show that there cannot be 3 degree three relations in the minimal generating set.

Theorem 4.2. *There is no AS-Ore extension generated in degree one with degree type $(1, 1, 1, 1, 2)$ and relation type $(2, 2, 2, 2, 2, 3, 3, 3)$.*

Proof. Assume to the contrary. As in the previous theorem, label the degree two variable x_1 and label the degree one variables so that $x_2 < x_3 < x_4 < x_5$ in the ordering. Then the general form of the possible degree two relations is the same as in the previous theorem. We will consider the different cases where the set of degree three leading terms leads to Gröbner bases with at least 3 degree three relations.

Case 1: The set of degree two LTs in \tilde{A} is $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_3\}$.

By the same argument as that in Theorem 4.1, since x_5x_2 is not a leading term, the relations with smaller leading terms must have coefficient in front of x_1 equal to zero and r_{52} must have a nonzero coefficient in front of the x_1 . Thus, $b_1 = e_1 = d_1 = 0$, $i_1 \neq 0$, there are 4 degree three relations in the Gröbner basis, and we have already seen in the proof of Theorem 4.2 that $x_5x_3x_2$ is an ambiguity that fails to resolve. Taking $x_5(x_3x_2) - (x_5x_3)x_2$ in \tilde{A} gives a new degree three relation with leading term $x_5x_2x_3$. When evaluating whether other degree three overlaps resolve, we should reduce them modulo this new relation, but this will not be necessary in the calculations that follow since $x_5x_2x_3$ is too small to affect the analysis of whether the overlaps resolve. Now consider the coefficient of $x_5x_2x_4$ in the following reduction of overlaps in \tilde{A} :

$$\begin{aligned}
x_5(x_4x_2) &= x_5(0x_1 + e_4x_2x_4 + e_6x_3x_3 + e_7x_3x_4 + [\text{smaller terms}]) \\
&= e_4x_5x_2x_4 + e_6x_5x_3x_3 + e_7x_5x_3x_4 + [\text{small}] \\
&= e_4x_5x_2x_4 + e_6(h_1x_1 + [\text{small}])x_3 + e_7(h_1x_1 + [\text{small}])x_4 + [\text{small}] \\
&= e_4x_5x_2x_4 + e_6\left(\frac{h_1}{i_1}x_5x_2 - [\text{small}]\right)x_3 + e_7\left(\frac{h_1}{i_1}x_5x_2 - [\text{small}]\right)x_4 + [\text{small}] \\
&= \left(e_4 + \frac{e_7h_1}{i_1}\right)x_5x_2x_4 + [\text{small}], \text{ while} \\
(x_5x_4)x_2 &= (g_1x_1 + [\text{smaller terms}])x_2 \\
&= \frac{g_1}{i_1}x_5x_2x_2 + [\text{small}]
\end{aligned}$$

$$\text{So } x_5(x_4x_2) - (x_5x_4)x_2 = \left(e_4 + \frac{e_7h_1}{i_1}\right)x_5x_2x_4 + [\text{smaller terms}].$$

Similarly, $x_5(x_4x_3) - (x_5x_4)x_3 = \left(d_4 + \frac{d_7h_1}{i_1}\right)x_5x_2x_4 + [\text{smaller terms}]$. From the relations

$$\begin{aligned}
r_{42} : x_4x_2 &= e_1x_1 + e_2x_2x_2 + e_3x_2x_3 + e_4x_2x_4 + e_6x_3x_3 + e_7x_3x_4 \text{ and} \\
r_{43} : x_4x_3 &= d_1x_1 + d_2x_2x_2 + d_3x_2x_3 + d_4x_2x_4 + d_6x_3x_3 + d_7x_3x_4,
\end{aligned}$$

we see that

$$\begin{aligned}\sigma_4(x_2) &= e_4x_2 + e_7x_3 \text{ and} \\ \sigma_4(x_3) &= d_4x_2 + d_7x_3.\end{aligned}$$

Thus, σ_4 is injective if and only if $\det \begin{bmatrix} e_4 & e_7 \\ d_4 & d_7 \end{bmatrix} \neq 0$. If we assume that both overlaps resolve, $e_4 = -\frac{e_7h_1}{i_1}$ and $d_4 = -\frac{d_7h_1}{i_1}$, so $\begin{vmatrix} e_4 & e_7 \\ d_4 & d_7 \end{vmatrix} = \begin{vmatrix} \frac{-e_7h_1}{i_1} & e_7 \\ -\frac{d_7h_1}{i_1} & d_7 \end{vmatrix} = 0$ and σ_4 is not injective, a contradiction. Thus, at least one of the two overlaps, $x_5x_4x_3$ or $x_5x_4x_2$, fails to resolve. In total, there are at least two overlaps that do not resolve so there are at most 2 degree three relations in the minimal generating set.

Case 2: The set of degree two LTs in \tilde{A} is $\{x_3x_2, x_4x_3, x_4x_2, x_5x_4, x_5x_2\}$. In this case, x_5x_3 is not a leading term and similar reasoning as that used in Theorem 4.1 allows us to conclude that $b_1 = e_1 = d_1 = i_1 = 0$, $h_1 \neq 0$, and there are 3 degree three relations in the Gröbner basis. We then compute $x_5(x_4x_2) - (x_5x_4)x_2 = e_7x_5x_3x_4 + [\text{smaller terms}]$ and $x_5(x_4x_3) - (x_5x_4)x_3 = d_7x_5x_3x_4 + [\text{smaller terms}]$. These computations can be done by hand by looking at the largest terms in the reduction, just as in the previous case, but we omit the details. The code used for all calculations in this proof is on the author's website [Ell, Section 3]. If these overlaps both resolve, $e_7 = 0$ and $d_7 = 0$, $\begin{vmatrix} e_4 & e_7 \\ d_4 & d_7 \end{vmatrix} = 0$, and σ_4 is not injective, which is a contradiction. Thus, one of these overlaps must fail to resolve and the minimal generating set has at most 2 relations of degree three.

Case 3: The set of degree two LTs in \tilde{A} is $\{x_3x_2, x_4x_3, x_5x_4, x_5x_3, x_5x_2\}$. In this case, x_4x_2 is not a leading term so $b_1 = 0$, $e_1 \neq 0$, and there are 3 degree three relations in the Gröbner basis. Then $x_4(x_3x_2) - (x_4x_3)x_2 = b_3x_4x_2x_3 + [\text{smaller terms}]$. This computation can be done by hand, but we omit the details here. If σ_3 is injective, b_3 cannot be zero. Thus, at least one overlap fails to resolve and the minimal generating set has at most 2 relations of degree three.

In all cases where the Gröbner basis has at least 3 degree three relations, we find that the minimal generating set has at most 2 degree three relations, so there is no AS-Ore extension with relation type $(2, 2, 2, 2, 2, 3, 3, 3)$. \square

There are algebras with relation types with 0, 1, and 2 degree three relations. We begin by considering what relation types can be realized by enveloping algebras.

Theorem 4.3. *There are enveloping algebras with degree type $(1, 1, 1, 1, 2)$ and relation type $(2, 2, 2, 2, 2)$ and $(2, 2, 2, 2, 2, 3, 3)$, but no enveloping algebra with relation type $(2, 2, 2, 2, 2, 2, 3)$.*

Proof. An enveloping algebra with such a degree type can be taken to have $x_5 > x_4 > x_3 > x_2 > x_1$ with $\deg(x_1)=2$ and is then defined by the relations

$$\begin{aligned} r_{21} : x_2x_1 &= x_1x_2 \\ r_{31} : x_3x_1 &= x_1x_3 \\ r_{41} : x_4x_1 &= x_1x_4 \\ r_{51} : x_5x_1 &= x_1x_5 \\ r_{32} : x_3x_2 &= b_1x_1 + x_2x_3 \\ r_{43} : x_4x_3 &= d_1x_1 + x_3x_4 \\ r_{42} : x_4x_2 &= e_1x_1 + x_2x_4 \\ r_{54} : x_5x_4 &= g_1x_1 + x_4x_5 \\ r_{53} : x_5x_3 &= h_1x_1 + x_3x_5 \\ r_{52} : x_5x_2 &= i_1x_1 + x_2x_5. \end{aligned}$$

These relations are homogeneous, overlaps resolve (see [Ell, Section 4]), σ_j is the identity for all $j \geq 1$, δ_j is linear for all $j \geq 1$ and this is generated in degree one if at least 1 of b_1 , d_1 , e_1 , g_1 , h_1 , or i_1 is nonzero. In this case, by Theorem 1.3, this is an enveloping algebra and it remains to find its relation type.

By the symmetry of the relations, we may assume without loss of generality that b_1 is nonzero and write $x_1 = \frac{-x_2x_3 + x_3x_2}{b_1}$. We can now view the algebra as \tilde{A} , something generated in degree one, so that the set of LTs of degree two relations is $\{x_4x_3, x_4x_2, x_5x_4, x_5x_3, x_5x_2\}$. Given this set of degree two LTs, we see from the analysis preceding Theorem 4.1 that the Gröbner basis of \tilde{A} has 2 degree three leading terms. Further, the degree three overlaps that come from this list of degree two LTs are $x_5x_4x_3$ and $x_5x_4x_2$. We calculate:

$$\begin{aligned} x_5(x_4x_3) - (x_5x_4)x_3 &= \left(\frac{g_1}{b_1} - \frac{e_1h_1}{b_1^2} + \frac{d_1i_1}{b_1^2}\right)x_2x_3x_3 \\ &+ \left(\frac{-2g_1}{b_1} + \frac{2e_1h_1}{b_1^2} - \frac{2d_1i_1}{b_1^2}\right)x_3x_2x_3 + \left(\frac{g_1}{b_1} - \frac{e_1h_1}{b_1^2} + \frac{d_1i_1}{b_1^2}\right)x_3x_3x_2, \text{ and} \\ x_5(x_4x_2) - (x_5x_4)x_2 &= \left(\frac{-g_1}{b_1} + \frac{e_1h_1}{b_1^2} - \frac{d_1i_1}{b_1^2}\right)x_2x_2x_3 \\ &+ \left(\frac{2g_1}{b_1} - \frac{2e_1h_1}{b_1^2} + \frac{2d_1i_1}{b_1^2}\right)x_2x_3x_2 + \left(\frac{-g_1}{b_1} + \frac{e_1h_1}{b_1^2} - \frac{d_1i_1}{b_1^2}\right)x_3x_2x_2. \end{aligned}$$

As usual, the details for these calculations are omitted here but included in the code posted online [Ell, Section 4]. If $g_1 = \frac{e_1h_1}{b_1} - \frac{d_1i_1}{b_1}$ then both of these overlaps resolve, the degree three relations in the Gröbner basis are independent of overlaps, and the relation type is $(2,2,2,2,3,3)$. Otherwise, these are two K -independent overlaps that fail to resolve and the minimal relation type is $(2,2,2,2,2)$. Since any enveloping algebra must have one of these two relation types, the theorem is proven. \square

Although there is no enveloping algebra with relation type $(2,2,2,2,3)$, it is possible to construct an AS-Ore extension with this type. Our process for doing this is similar to that used to find the AS-Ore extension of degree type $(1,1,2,3,5)$ in Theorem 3.2. We use Theorem 1.5 to write general relations for the extension and

a mathematical program to evaluate the possible values of coefficients that make it so that those relations satisfy the diamond condition. This problem is generally too complex for the computer to handle, so we can set some coefficients equal to zero to simplify the process. In this case, we also have to determine how many degree three relations the Gröbner basis of the algebra generated in degree one has, as well as whether these relations are part of the minimal generating set as opposed to consequences of overlaps that do not resolve.

Theorem 4.4. *There is an AS-Ore extension with degree type $(1,1,1,1,2)$ and relation type $(2,2,2,2,2,3)$.*

Proof. Consider the algebra defined by the relations

$$\begin{aligned} r_{21} : x_2x_1 &= x_1x_2 \\ r_{32} : x_3x_2 &= x_1 + x_2x_2 - x_2x_3 \\ r_{31} : x_3x_1 &= x_1x_3 \\ r_{43} : x_4x_3 &= x_2x_2 - x_3x_4 \\ r_{42} : x_4x_2 &= -x_2x_4 \\ r_{41} : x_4x_1 &= x_1x_4 \\ r_{54} : x_5x_4 &= -x_3x_3 - x_4x_5 \\ r_{53} : x_5x_3 &= x_3x_3 - x_3x_5 \\ r_{52} : x_5x_2 &= -x_2x_5 + x_3x_3 \\ r_{51} : x_5x_1 &= x_1x_5. \end{aligned}$$

Taking $x_5 > \dots > x_1$, the leading terms are as presented and all overlaps resolve [Ell, Section 5]. Taking $\deg(x_1) = 2$ and $\deg(x_i) = 1$, $2 \leq i \leq 5$, the relations are homogeneous. Thus by Theorem 1.6, this is an Ore extension. It is also generated in degree one since r_{32} can be used to express x_1 in terms of the degree one generators. Finally, $\sigma_j(x_i) = \pm 1$ for all $1 \leq i < j \leq 5$ so these maps are injective and this is an AS-Ore extension.

We can solve r_{32} for x_1 and then view the algebra as \tilde{A} , generated in degree one. From the analysis preceding Theorem 4.1, the Gröbner basis of \tilde{A} has 2 degree three relations and it remains to show that exactly one of these is a consequence of an overlap that fails to resolve. We compute

$$x_5(x_4x_3) - (x_5x_4)x_3 = x_2x_2x_3 - x_2x_3x_3 - x_3x_2x_2 + x_3x_3x_2$$

so this overlap never resolves. Reducing the remaining overlap modulo this additional relation,

$$x_5(x_4x_2) - (x_5x_4)x_2 = 0.$$

Thus, 1 of the degree three relations in the Gröbner basis is minimal and the relation type of this algebra is $(2,2,2,2,2,3)$. \square

5. A CLASSIFICATION OF RELATION TYPES FOR AS-ORE EXTENSIONS WITH 3 DEGREE ONE GENERATORS

We now begin the process of classifying the relation types of AS-Ore extensions with 3 degree one generators. Again, by the symmetry of the free resolution, this also provides us with all the information we need to classify all possible resolution

types of such algebras. We recall that, by Theorem 2.4, there are two possible degree types for algebras with 3 generators: $(1,1,1,2,2)$ and $(1,1,1,2,3)$.

Theorem 5.1. *An AS-Ore extension with degree type $(1,1,1,2,2)$ has relation type $(2,3,3,3,3,3)$.*

Proof. By Equation (0.1), a dimension 5 AS-Ore extension generated in degree one by 3 generators has free resolution

$$0 \rightarrow A(-l) \rightarrow A(-l+1)^b \rightarrow \bigoplus_{i=1}^n A(-l+a_i) \rightarrow \bigoplus_{i=1}^n A(-a_i) \rightarrow A(-1)^b \rightarrow A \rightarrow K \rightarrow 0$$

where $b=3$ since there are 3 degree one generators, n represents the number of relations in the minimal generating set and a_i represents the degree of the i th relation. This algebra has Hilbert series

$$h_A(t) = \frac{1}{q(t)} \text{ where } q(t) = 1 - 3t + \sum_{i=1}^n t^{a_i} - \sum_{i=1}^n t^{l-a_i} + 3t^{l-1} - t^l.$$

On the other hand, an AS-Ore extension with degree type $(1,1,1,2,2)$ has Hilbert series

$$h_A(t) = \frac{1}{(1-t)^3(1-t^2)^2}, \text{ so } q(t) = 1 - 3t + t^2 + 5t^3 - 5t^4 - t^5 + 3t^6 - t^7.$$

Assume $a_1 \leq \dots \leq a_n$. Then $l = 7$, $a_1 = 2$, $a_i = 3$ for $2 \leq i \leq 6$ (there are 5 degree three relations), and if there are any other minimal relations, they must cancel in the expression $\sum_{i=1}^n t^{a_i} - \sum_{i=1}^n t^{l-a_i}$ since they do not appear in the second equation for $q(t)$. This would mean either that $a_i = l - a_i$ (which is impossible since l is odd) or that there are at least two additional relations, a_i and a_j with $a_i + a_j = l$. Although we are interested in the minimal generating set of \tilde{A} , the algebra generated in degree one, we note that any minimal relations of \tilde{A} must come from the original relations of the algebra viewed as an Ore extension, A . The possible relations for A are described in the presentation of an Ore extension given by Theorem 1.5 and an Ore extension with degree type $(1,1,1,2,2)$ can only have relations of degrees two, three, or four. Thus, if there are additional minimal relations that cancel in the Hilbert series, they must be of degree three and four (since something of degree two or lower could only cancel if there were also a relation of degree five or higher, and we know that an Ore extension with this degree type has no such relations which are minimal).

If we label the degree one generators with the order $x_3 < x_4 < x_5$, the algebra, when viewed as \tilde{A} , will have 1 degree two relation with leading term x_5x_4 , x_5x_3 , or x_4x_3 . (The remaining 2 degree two relations will be used to write the x_1 and x_2 in terms of the degree one generators.) If \tilde{A}_2 denotes the monomial algebra generated in degree one that has one of x_5x_4 , x_5x_3 , or x_4x_3 as a relation and \tilde{A} denotes the AS-Ore extension with degree type $(1,1,1,2,2)$ viewed as an algebra generated in degree one, then $h_{\tilde{A}_2}(t) - h_{\tilde{A}}(t) = 5t^3 + O(t^4)$ [Ell, Section 6], so there can only be 5 degree three relations in the Gröbner basis. Thus, the only relation type for an AS-Ore extension with degree type $(1,1,1,2,2)$ is $(2,3,3,3,3,3)$. \square

Theorem 5.2. *There is an enveloping algebra with degree type $(1,1,1,2,2)$ and relation type $(2,3,3,3,3,3)$.*

Proof. Consider the algebra defined by relations

$$\begin{aligned}
r_{21} : x_2x_1 &= x_1x_2 \\
r_{32} : x_3x_2 &= x_2x_3 \\
r_{31} : x_3x_1 &= x_1x_3 \\
r_{42} : x_4x_2 &= x_2x_4 \\
r_{41} : x_4x_1 &= x_1x_4 \\
r_{52} : x_5x_2 &= x_2x_5 \\
r_{51} : x_5x_1 &= x_1x_5 \\
r_{43} : x_4x_3 &= x_1 + x_3x_4 \\
r_{54} : x_5x_4 &= x_2 + x_4x_5 \\
r_{53} : x_5x_3 &= x_3x_5.
\end{aligned}$$

Assigning $(x_5, x_4, x_3, x_2, x_1)$ degrees $(1, 1, 1, 2, 2)$, these relations are homogeneous and it can be verified by hand or computer that all overlaps resolve, so this is an Ore extension by Theorem 1.6. Additionally, for all $1 \leq i < j \leq 5$, $\sigma_j(x_i)$ is the identity and $\delta_j(x_i)$ is linear, so this is an enveloping algebra by Theorem 1.3. It is also generated in degree one. Another quick check in Mathematica shows that the relation type is $(2, 3, 3, 3, 3, 3)$ [Ell, Section 7], although the analysis from Theorem 5.1 already indicates that this has to be the case since this is the only possible relation type for an AS-Ore extension with this degree type. \square

Theorem 5.3. *An AS-Ore extension with degree type $(1, 1, 1, 2, 3)$ has relation type $(2, 2, 3)$ or $(2, 2, 3, 4)$.*

Proof. Following the logic of Theorem 5.1, we know from the free resolution of the algebra that

$$h_A(t) = \frac{1}{q(t)} \text{ where } q(t) = 1 - 3t + \sum_{i=1}^n t^{a_i} - \sum_{i=1}^n t^{l-a_i} + 3t^{l-1} - t^l.$$

On the other hand, an AS-Ore extension with degree type $(1, 1, 1, 2, 3)$ has Hilbert series

$$h_A(t) = \frac{1}{(1-t)^3(1-t^2)(1-t^3)}, \text{ so } q(t) = 1 - 3t + 2t^2 + t^3 - t^5 - 2t^6 + 3t^7 - t^8.$$

So $l = 8$, $a_1 = a_2 = 2$, $a_3 = 3$, and if there are any other minimal relations of \tilde{A} , generated in degree one, they must cancel. In this case we would have $a_i = l - a_j$ (where it is possible that $i = j$). This means that there may possibly be a pair of minimal relations of degree three and five (there cannot be more than one such pair since there is only 1 degree five relation in the Ore extension) and there may possibly be up to 3 relations of degree four (since the Ore extension A has 3 relations of degree four). There cannot be any additional minimal relations of degree two since they would need to cancel with something of degree six and there are no minimal relations of degree six for an AS-Ore extension with this degree type. To conclude that the relation type is either $(2, 2, 3)$ or $(2, 2, 3, 4)$, it remains to show that there are not 2 independent relations of degree three (and so no additional pair of

relations of degree three and five) and that there is at most 1 minimal relation of degree four.

We can write relations based on the fact that this is an Ore extension. Without loss of generality, label the degree one generators so that $x_3 < x_4 < x_5$. We will let x_2 be the degree two variable and x_1 will be the degree three variable, but we make no claim about when these variables are adjoined. (Thus possible orders include $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_3 < x_2 < x_4 < x_1 < x_5$, amongst many others.) We do note, however, that for A to be generated in degree one, x_1 must not be adjoined last (since it must be adjoined by the time it appears in a relation with leading term of the form $x_j x_i$ and hence must be adjoined before x_j), and (similarly) x_2 must not be adjoined last. Then by the choice of ordering of our degree one variables, x_4 and x_3 are both adjoined before x_5 , so x_5 is always added last in the iterated Ore extension.

Using the ordering of the degree one variables, we can write the degree two relations. We note that the only thing that could change in the relations below that depends on the order in which the variables is adjoined is that d_1 must be zero if x_2 is added after x_4 , since r_{43} should only involve variables that have been added by the time x_4 is adjoined. The degree two relations are:

$$\begin{aligned} r_{43} : x_4 x_3 &= d_1 x_2 + d_2 x_3 x_3 + d_3 x_3 x_4 \\ r_{53} : x_5 x_3 &= h_1 x_2 + h_2 x_3 x_3 + h_3 x_3 x_4 + h_4 x_3 x_5 + h_5 x_4 x_4 + h_6 x_4 x_5 \\ r_{54} : x_5 x_4 &= g_1 x_2 + g_2 x_3 x_3 + g_3 x_3 x_4 + g_4 x_3 x_5 + g_5 x_4 x_4 + g_6 x_4 x_5. \end{aligned}$$

We wish to repeat this process for the degree three relations. The list of possible degree three monomials that can appear on the right side of a relation, given the ordering $x_3 < x_4 < x_5$, $x_1 < x_5$, $x_2 < x_5$, is:

$$\{x_1, x_2 x_3, x_3 x_2, x_2 x_4, x_4 x_2, x_2 x_5, x_3 x_3 x_3, x_3 x_3 x_4, x_3 x_3 x_5, x_3 x_4 x_4, x_3 x_4 x_5, x_4 x_4 x_4\}.$$

Since the ordering of the variables is not known, the leading terms of the degree 3 relations can vary, as can the other allowable monomials. We will write fully general versions of the relations for each possible leading term. It is appropriate to use r_{32a} below when $x_3 > x_2$ and r_{32b} when $x_2 > x_3$. Similarly, r_{42a} applies when $x_4 > x_2$, and r_{42b} when $x_2 > x_4$.

From Theorem 1.5, x_j must occur only to the first power in the relation with leading term $x_j x_i$ and x_k should not appear for any $x_k > x_j$. For example in r_{32b} below, the leading term implies that $x_2 > x_3$. If $x_2 > x_4 > x_3$ then the monomial $x_4 x_2$ may appear. Otherwise, $x_4 > x_2 > x_3$ and the monomial $x_2 x_4$ does not appear since x_4 has not yet been adjoined in the Ore extension. The most general possible degree three relations are:

$$\begin{aligned} r_{32a} : -b_0 x_3 x_2 &= b_1 x_1 + b_2 x_2 x_3 \\ r_{32b} : -b_2 x_2 x_3 &= b_1 x_1 + b_0 x_3 x_2 + b_3 x_4 x_2 + b_4 x_3 x_3 x_3 + b_5 x_3 x_3 x_4 + b_6 x_3 x_4 x_4 \\ &\quad + b_7 x_4 x_4 x_4 \end{aligned}$$

$$\begin{aligned}
r_{42}a : \quad & -e_0x_4x_2 = e_1x_1 + e_2x_2x_3 + e_3x_2x_4 + e_4x_3x_2 + e_5x_3x_3x_3 + e_6x_3x_3x_4 \\
r_{42}b : \quad & -e_3x_2x_4 = e_1x_1 + e_0x_4x_2 + e_4x_3x_2 + e_5x_3x_3x_3 + e_6x_3x_3x_4 + e_7x_3x_4x_4 \\
& + e_8x_4x_4x_4 \\
r_{52} : \quad & x_5x_2 = i_1x_1 + i_2x_2x_3 + i_3x_2x_4 + i_4x_3x_2 + i_5x_4x_2 + i_6x_2x_5 \\
& + i_7x_3x_3x_3 + i_8x_3x_3x_4 + i_9x_3x_3x_5 + i_{10}x_3x_4x_4 + i_{11}x_3x_4x_5 \\
& + i_{12}x_4x_4x_4 + i_{13}x_4x_4x_5.
\end{aligned}$$

Note that not all of these coefficients can be nonzero. For example, if $x_2 < x_4$ then $i_5 = 0$ while if $x_4 < x_2$, $i_3 = 0$. These do, however, capture all possible terms that could occur in the Ore relations.

The goal now is to view this as \tilde{A} , generated in degree one, and to try to identify the possible degrees of relations that can occur. The proof depends on which relations are used to solve for x_2 . Without loss of generality, we will solve for x_1 and x_2 using the relation with smallest leading term so that the leading terms of the remaining degree two and three relations are easily identifiable.

Case 1: x_2 comes from r_{43} .

In this case, the degree two leading terms are x_5x_4 and x_5x_3 . If \tilde{A}_2 is the monomial algebra with these leading terms then $h_{\tilde{A}_2} - h_{\tilde{A}} = t^3 + O(t^4)$ so there is 1 degree three relation in the Gröbner basis of \tilde{A} and so at most (and exactly) 1 degree three relation in the minimal generating set, as we wished to show. It remains to show that there is no more than 1 degree four relation in the minimal generating set.

Since x_2 comes from r_{43} , x_2 certainly appears in the relation r_{43} and so must have been adjoined before x_4 . This means that $x_2 < x_4$ in the order which in turn implies that we should use the relation $r_{42}a$, that e_0 is not zero since it is the leading term of the expression, and that $b_3 = b_5 = b_6 = b_7 = 0$. We can begin to view the algebra as generated in degree one by solving r_{43} to get $x_2 = \frac{1}{d_1}(x_4x_3 - d_3x_3x_4 - d_2x_3x_3)$. We can substitute this into the other relations and move all terms to the right side of the equation:

$$\begin{aligned}
r_{32a} : 0 &= b_0x_3x_2 + b_1x_1 + b_2x_2x_3 \\
&= b_0x_3 \frac{1}{d_1}(x_4x_3 - d_3x_3x_4 - d_2x_3x_3) + b_1x_1 \\
&\quad + b_2 \frac{1}{d_1}(x_4x_3 - d_3x_3x_4 - d_2x_3x_3)x_3 \\
&= b_1x_1 + \frac{b_2}{d_1}x_4x_3x_3 + [\text{smaller terms}] \\
r_{32b} : 0 &= b_2x_2x_3 + b_1x_1 + b_0x_3x_2 + 0x_4x_2 + b_4x_3x_3x_3 + 0x_3x_3x_4 + 0x_3x_4x_4 \\
&\quad + 0x_4x_4x_4 \\
&= b_2 \frac{1}{d_1}(x_4x_3 - d_3x_3x_4 - d_2x_3x_3)x_3 + b_1x_1 + b_4x_3x_3x_3 \\
&= b_1x_1 + \frac{b_2}{d_1}x_4x_3x_3 + [\text{smaller terms}].
\end{aligned}$$

We note that the largest term appearing in the relation for r_{32} is now independent of which version of the relation we use and, repeating the substitution for x_2 in r_{42a} ,

we can write

$$\begin{aligned} r_{32} : 0 &= b_1 x_1 + \frac{b_2}{d_1} x_4 x_3 x_3 + [\text{smaller terms}] \\ r_{42} : 0 &= e_1 x_1 + \frac{e_0}{d_1} x_4 x_4 x_3 + [\text{smaller terms}]. \end{aligned}$$

From the original relations, we observe that b_2 is not zero: either $x_2 > x_3$ in the ordering and b_2 is the LT of r_{32} or $x_3 > x_2$ and $\sigma_3(x_2) = b_2 x_2$ and so $b_2 \neq 0$ by the injectivity of σ_3 .

If b_1 is not zero, we may solve r_{32} for x_1 and, even after substituting this value into r_{42} , the leading term of r_{42} in \tilde{A} will be $x_4 x_4 x_3$. If b_1 is zero then $x_4 x_3 x_3$ will be a leading term in \tilde{A} . If \tilde{A}_3 is the monomial algebra with LTs $\{x_5 x_4, x_5 x_3, x_4 x_3 x_3\}$ or $\{x_5 x_4, x_5 x_3, x_4 x_4 x_3\}$ then $h_{\tilde{A}_3} - h_{\tilde{A}} = t^4 + O(t^5)$. So the Gröbner basis of \tilde{A} has 1 degree four relation and so at there is at most 1 degree four relation in the minimal generating set, as we wished to show.

Case 2: x_2 comes from r_{54} .

In this case, the degree two leading terms are $x_5 x_3$ and $x_4 x_3$, $d_1 = g_1 = 0$ (or else x_2 would come from r_{43} or r_{53}), and $h_{\tilde{A}_2} - h_A = t^3 + O(t^4)$. This means there is at most 1 degree three relation in the minimal generating set and it remains to show that there is also at most 1 degree four relation. From r_{54} , we can write $x_2 = \frac{1}{g_1}(x_5 x_4 + [\text{smaller terms}])$. As in the first case, we can substitute this value into the degree three terms to get that the highest terms in the degree three relations of interest are known, independent of which version of r_{42} we use:

$$\begin{aligned} r_{42} : 0 &= e_1 x_1 + \frac{e_3}{g_1} x_5 x_4 x_4 + [\text{smaller terms}] \\ r_{52} : 0 &= i_1 x_1 - \frac{1}{g_1} x_5 x_5 x_4 + [\text{smaller terms}]. \end{aligned}$$

Note that e_3 is not zero: either $x_4 < x_2$ and it e_3 the leading coefficient of r_{42b} , or $x_2 < x_4$ and from r_{42a} , $\sigma_4(x_2) = -\frac{e_3}{e_0} x_2 - \frac{e_6}{e_0} x_3 x_3$. In this case, since $\sigma_4(x_3) = d_3 x_3$, $\sigma_4(x_2 + \frac{e_6}{e_0 d_3^2} x_3 x_3) = -\frac{e_3}{e_0} x_2 - \frac{e_6}{e_0} x_3 x_3 + \frac{e_6}{e_0} x_3 x_3$. By the injectivity of σ_4 , $e_3 \neq 0$.

If e_1 is not zero, we may solve r_{42} for x_1 and, even after substituting this value into r_{52} , the LT of r_{52} will be $x_5 x_5 x_4$. If e_1 is zero then $x_5 x_4 x_4$ will be a leading term in \tilde{A} . In either case, we calculate that $h_{\tilde{A}_3} - h_{\tilde{A}} = t^4 + O(t^5)$, which means that there is at most 1 degree four relation in the minimal generating set, as we wished to prove.

Case 3: x_2 comes from r_{53} .

In this case, the degree two LTs are $x_5 x_4$ and $x_4 x_3$, $d_1 = 0$, and $h_{\tilde{A}_2} - h_{\tilde{A}} = 2t^3 - O(t^4)$. We can solve r_{53} to get that $x_2 = \frac{1}{h_1}(x_5 x_3 - h_6 x_4 x_5 + [\text{smaller terms}])$. We can rewrite the remaining degree two relations after substituting the value of x_2 into the equations:

$$\begin{aligned} r_{43} : x_4 x_3 &= 0 x_2 + d_3 x_3 x_4 + [\text{smaller terms}] \\ r_{54} : x_5 x_4 &= g_1 \frac{1}{h_1} (x_5 x_3 + [\text{smaller terms}]) + g_6 x_4 x_5 + [\text{smaller terms}]. \end{aligned}$$

We can then compute the degree three overlap in \tilde{A} .

$$\begin{aligned} (x_5x_4)x_3 - x_5(x_4x_3) &= \frac{g_1}{h_1}(x_5x_3 - h_6x_4x_5 + [\text{small}])x_3 - x_5(d_3x_3x_4 + d_2x_3x_3) \\ &= -d_3x_5x_3x_4 + \left(\frac{g_1}{h_1} - d_2\right)x_5x_3x_3 + [\text{small}]. \end{aligned}$$

Since d_3 is not zero by the injectivity of σ_3 , this overlap does not resolve and we know that there is a relation, $x_5x_3x_4 = \frac{g_1 - d_2h_1}{h_1}x_5x_3x_3 + [\text{smaller terms}]$ in the Gröbner basis of \tilde{A} . We conclude that at most 1 of the 2 degree three relations in the Gröbner basis can be minimal and it remains to show that there is at most 1 minimal relation of degree four.

Again substituting the value of x_2 , we may re-examine the degree three relations and note that, as with the first case, the LT of r_{32} is independent of which version we use:

$$\begin{aligned} r_{32} : 0 &= b_1x_1 + \frac{b_2}{h_1}x_5x_3x_3 + [\text{smaller terms}] \\ r_{52} : 0 &= i_1x_1 - \frac{1}{h_1}x_5x_5x_3 + [\text{smaller terms}]. \end{aligned}$$

By the same analysis as in case 1, $b_2 \neq 0$. If $b_1 = 0$ then the leading term of r_{32} in \tilde{A} is $x_5x_3x_3$. If \tilde{A}_3 is the monomial algebra with LTs $\{x_5x_4, x_4x_3, x_5x_3x_3, x_5x_3x_4\}$ then $h_{\tilde{A}_3} - h_{\tilde{A}} = t^4 + O(t^5)$ and there is at most 1 degree 4 relation in the minimal generating set as desired.

If b_1 is not zero then r_{32} may be solved for x_1 and, even after substituting the value of x_1 into the relation, the leading term of r_{52} in \tilde{A} is $x_5x_5x_3$. Thus, the 2 relations of degree three in the Gröbner basis have LTs $x_5x_3x_4$ and $x_5x_5x_3$ and in this case, $h_{\tilde{A}_3} - h_{\tilde{A}} = 2t^4 + O(t^5)$. We can also compute the degree four overlap:

$$\begin{aligned} (x_5x_3x_4)x_3 - x_5x_3(x_4x_3) &= \left(\frac{g_1 - d_2h_1}{d_3h_1}x_5x_3x_3 + [\text{small}]\right)x_3 \\ &\quad - x_5x_3(d_3x_3x_4 + [\text{small}]) \\ &= -d_3x_5x_3x_3x_4 + [\text{small}]. \end{aligned}$$

We note that $x_5x_3x_3x_4$ cannot be reduced in \tilde{A}_3 and d_3 is not zero so this overlap fails to resolve. In total, we have found that the Gröbner basis of \tilde{A} has 2 degree four relations, at least one of which is not minimal.

Thus, in all cases we have shown that \tilde{A} has at most 1 degree three and at most 1 degree four relation in the minimal generating set, which means that the relation type must be either (2,2,3), or (2,2,3,4). \square

Theorem 5.4. *There is an enveloping algebra with degree type (1,1,1,2,3) and relation type (2,2,3,4), but not one with relation type (2,2,3).*

Proof. An enveloping algebra can be taken to have $x_5 > x_4 > x_3 > x_2 > x_1$ with $\deg(x_2) = 2$ and $\deg(x_1) = 3$ and is then defined by the relations

$$\begin{aligned}
r_{21} : x_2x_1 &= x_1x_2 \\
r_{31} : x_3x_1 &= x_1x_3 \\
r_{41} : x_4x_1 &= x_1x_4 \\
r_{51} : x_5x_1 &= x_1x_5 \\
r_{32} : x_3x_2 &= b_1x_1 + x_2x_3 \\
r_{42} : x_4x_2 &= e_1x_1 + x_2x_4 \\
r_{52} : x_5x_2 &= i_1x_1 + x_2x_5 \\
r_{43} : x_4x_3 &= d_1x_2 + x_3x_4 \\
r_{54} : x_5x_4 &= g_1x_2 + x_4x_5 \\
r_{53} : x_5x_3 &= h_1x_2 + x_3x_5.
\end{aligned}$$

By construction we have that for all $1 \leq i < j \leq 5$, $\sigma_j(x_i)$ is the identity and $\delta_j(x_i)$ is linear for all i . All overlaps resolve, except for $x_5(x_4x_3) - (x_5x_4)x_3 = (b_1g_1 - e_1h_1 + d_1i_1)x_1$, so we will have to choose values of coefficients which make this expression zero. Additionally, to be generated in degree one, we must have that at least 1 of b_1, e_1, i_1 and at least 1 of d_1, g_1, h_1 is nonzero. If this happens, then by Theorem 1.3, this is an enveloping algebra.

We can now solve for x_1 and x_2 to view this as \tilde{A} and analyze the possible degrees of minimal relations. As the process for these computations is quite similar to that seen in previous theorems, we will omit most of the details. Our goal is to show that the relation type is always (2,2,3,4). By the symmetry of the relations, we may assume that b_1 is the coefficient that is not zero.

Case 1: d_1 nonzero.

From the overlap in A , $x_5(x_4x_3) - (x_5x_4)x_3 = (g_1b_1 - e_1h_1 + i_1d_1)x_1$, we conclude that $g_1 = \frac{e_1h_1 - i_1d_1}{b_1}$. Solving r_{43} and r_{32} for x_2 and x_1 and substituting these into the remaining relations to view the algebra as generated in degree one, we find that there are degree two relations in \tilde{A} with LTs x_5x_3 and x_5x_4 and a degree three relation from r_{42} with LT $x_4x_4x_3$. Reduced modulo these relations, r_{51} then has LT $x_4x_3x_3x_3$ and it remains to show that this is minimal in \tilde{A} . The only degree four overlap, given these LTs, is $(x_5x_4)x_4x_3 - x_5(x_4x_4x_3) = 0$. Details for computations in this proof are available online [Ell, Section 9]. Since this overlap resolves, the degree four relation is independent. By Theorem 5.3, this means that the relation type must be (2,2,3,4). In particular, we have now shown that there is an enveloping algebra with this relation type. It remains to show that (2,2,3) is never a possible relation type.

Case 2: $d_1 = 0$ and h_1 nonzero.

From the overlap $x_5(x_4x_3) - (x_5x_4)x_3 = (-e_1h_1 + g_1b_1)x_1$ we conclude that $e_1 = \frac{g_1b_1}{h_1}$. Solving r_{53} and r_{32} for x_2 and x_1 and substituting these into the remaining relations to view the algebra as generated in degree 1, we find that there

are degree two relations in \tilde{A} with LTs x_4x_3 and x_5x_4 , a degree three overlap that fails to resolve with LT $x_5x_3x_4$, and a degree three relation from r_{52} with LT $x_5x_5x_3$. The monomial algebra with these LTs has 2 degree four relations so it remains to show that exactly one such relation is the consequence of an overlap that does not resolve. One such overlap is $x_5x_3(x_4x_3) - (x_5x_3x_4)x_3 = \frac{g_1}{h_1}x_3x_5x_3x_3 - x_4x_5x_3x_3 - \frac{g_1}{h_1}x_5x_3x_3x_3 + x_5x_3x_3x_4$ and so fails to resolve. The remaining degree four overlap, when reduced modulo the degree two and three relations together with this new relation, is $x_5(x_5x_3)x_4 - (x_5x_5x_3)x_4 = 0$. Thus, there is 1 independent degree four relation and the relation type is $(2,2,3,4)$.

Case 3: $d_1 = 0$, $h_1 = 0$, and g_1 nonzero.

Recall that, by the symmetry of the relations, we have assumed that $b_1 \neq 0$, and that $g_1 \neq 0$ if $d_1 = 0$, $h_1 = 0$, and A is generated in degree one. From the overlap in A , $x_5(x_4x_3) - (x_5x_4)x_3 = b_1g_1x_1$, we conclude that there is no enveloping algebra with coefficients with these values since the overlaps of the original Ore relations fail to resolve.

Thus in all cases, the only possible relation type is $(2,2,3,4)$ and so this is the only relation type of an enveloping algebra with variables of degrees $(1,1,1,2,3)$. \square

Theorem 5.5. *There is an AS-Ore extension with relation type $(2,2,3)$.*

Proof. Consider the algebra defined by the relations

$$\begin{aligned} r_{21} : x_2x_1 &= -x_1x_2 \\ r_{32} : x_3x_2 &= x_2x_3 \\ r_{31} : x_3x_1 &= x_1x_3 \\ r_{43} : x_4x_3 &= x_3x_4 \\ r_{42} : x_4x_2 &= x_1 + x_2x_4 \\ r_{54} : x_5x_4 &= x_2 + x_4x_5 \\ r_{53} : x_5x_3 &= x_3x_5 + x_4x_4 \\ r_{52} : x_5x_2 &= x_2x_5 \\ r_{51} : x_5x_1 &= x_1x_5. \end{aligned}$$

Assigning $(x_5, x_4, x_3, x_2, x_1)$ degrees $(1,1,1,2,3)$, these relations are homogeneous. Using the order $x_5 > x_4 > x_3 > x_2 > x_1$ so that the relations are as presented, all overlaps resolve [Ell, Section 10]. Hence, this is an Ore extension by Theorem 1.6. It is also generated in degree one. For all $1 \leq i < j \leq 5$, $\sigma_j(x_i) = \pm 1$ so the σ_j are automorphisms and this algebra is AS-Ore.

We can view this algebra as \tilde{A} , generated in degree one, by solving r_{54} and r_{42} for x_2 and x_1 and plugging these values into the relations. The remaining degree two relations in \tilde{A} have LTs x_5x_3 and x_4x_3 and there is a degree three relation with LT $x_5x_5x_4$. Reduced modulo these relations, r_{31} and r_{51} become 0 while $r_{41} = -x_4x_4x_4x_5 + x_4x_4x_5x_4 + x_4x_5x_4x_4 - x_5x_4x_4x_4$. So there is 1 degree four relation in the Gröbner basis (which we already knew from the Hilbert series analysis of Theorem 5.3) and it remains to show that this is not minimal. We compute the overlap $x_5x_5(x_4x_3) - (x_5x_5x_4)x_3 = -x_4x_4x_4x_5 + x_4x_4x_5x_4 + x_4x_5x_4x_4 - x_5x_4x_4x_4$.

Thus, the degree four relation is a consequence of an overlap that fails to resolve and the relation type is $(2,2,3)$. \square

REFERENCES

- [AS] Michael Artin and William F. Schelter. Graded algebras of global dimension 3. *Adv. in Math.*, 66(2):171–216, 1987.
- [AST] Michael Artin, William Schelter, and John Tate. Quantum deformations of GL_n . *Comm. Pure Appl. Math.*, 44(8-9):879–895, 1991.
- [ATVdB1] M. Artin, J. Tate, and M. Van den Bergh. Some algebras associated to automorphisms of elliptic curves. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 33–85. Birkhäuser Boston, Boston, MA, 1990.
- [ATVdB2] M. Artin, J. Tate, and M. Van den Bergh. Modules over regular algebras of dimension 3. *Invent. Math.*, 106(2):335–388, 1991.
- [Ber] George M. Bergman. The diamond lemma for ring theory. *Adv. in Math.*, 29(2):178–218, 1978.
- [Coh] Paul Moritz Cohn. *Algebra. Vol. 2*. John Wiley & Sons, London-New York-Sydney, 1977. With errata to Vol. I.
- [Ell] Susan Elle. Personal website <http://www.math.ucsd.edu/~selle/research.html>, 2015.
- [FV] Gunnar Fløystad and Jon Eivind Vatne. Artin-Schelter regular algebras of dimension five. In *Algebra, geometry and mathematical physics*, volume 93 of *Banach Center Publ.*, pages 19–39. Polish Acad. Sci. Inst. Math., Warsaw, 2011.
- [GW] K. R. Goodearl and R. B. Warfield, Jr. *An introduction to noncommutative Noetherian rings*, volume 61 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 2004.
- [Hum] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Springer-Verlag, New York-Berlin, 1972. Graduate Texts in Mathematics, Vol. 9.
- [LPWZ] D.-M. Lu, J. H. Palmieri, Q.-S. Wu, and J. J. Zhang. Regular algebras of dimension 4 and their A_∞ -Ext-algebras. *Duke Math. J.*, 137(3):537–584, 2007.
- [Rog] D. Rogalski. An introduction to Noncommutative Projective Geometry. *ArXiv e-prints 1403.3065*, March 2014.
- [RZ] D. Rogalski and J. J. Zhang. Regular algebras of dimension 4 with 3 generators. In *New trends in noncommutative algebra*, volume 562 of *Contemp. Math.*, pages 221–241. Amer. Math. Soc., Providence, RI, 2012.
- [WW] S.-Q. Wang and Q.-S. Wu. A class of AS-regular algebras of dimension five. *J. Algebra*, 362:117–144, 2012.
- [ZL] G.-S. Zhou and D.-M. Lu. Artin-Schelter regular algebras of dimension five with two generators. *J. Pure Appl. Algebra*, 218(5):937–961, 2014.
- [ZZ1] James J. Zhang and Jun Zhang. Double Ore extensions. *J. Pure Appl. Algebra*, 212(12):2668–2690, 2008.
- [ZZ2] James J. Zhang and Jun Zhang. Double extension regular algebras of type (14641). *J. Algebra*, 322(2):373–409, 2009.

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